

Linear Separation of Total Dominating Sets in Graphs*

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Abstract

A total dominating set in a graph is a set of vertices such that every vertex of the graph has a neighbor in the set. We introduce and study graphs that admit non-negative real weights associated to their vertices so that a set of vertices is a total dominating set if and only if the sum of the corresponding weights exceeds a certain threshold. We show that these graphs, which we call total domishold graphs, form a non-hereditary class of graphs properly containing the classes of threshold graphs and the complements of domishold graphs. We present a polynomial time recognition algorithm of total domishold graphs, and obtain partial results towards a characterization of graphs in which the above property holds in a hereditary sense. Our characterization in the case of split graphs is obtained by studying a new family of hypergraphs, defined similarly as the Sperner hypergraphs, which might be of independent interest.

Keywords: total domination, total domishold graph, split graph, dually Sperner hypergraph, threshold hypergraph

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1 Introduction and Background

In many applications of graph theory, it is of crucial importance to be able to efficiently detect various kinds of substructures in graphs (matchings, cliques, stable sets, dominating sets, etc.). Whenever the respective computational problem turns out to be intractable, a

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possible approach is to identify restrictions on input instances under which the problem can still be solved efficiently. One generic framework for describing a kind of such restrictions is the following: Given a graph G , does G admit non-negative real weights on its vertices (or edges, depending on the problem) and a set T of real numbers such that a subset X of its vertices (or edges) has property P if and only if the sum of the weights of elements of X belongs to T ? Property P can denote any of the desired substructures we are looking for, such as matchings, cliques, stable sets, dominating sets, etc. If weights as above are integer and given with the graph, and the set T is given by a membership oracle, then a dynamic programming algorithm can be employed to find a subset with property P of either maximum or minimum cost (according to some given cost function on the vertices) in $O(nM)$ time and with M calls of the membership oracle, where n is the number of vertices (or edges) of G and M is a given upper bound for T [17].

Typically, the advantages of the above framework depend both on the choice of property P and on the constraints (if any) imposed on the structure of the set T . For example, if P denotes the property of being a stable (independent) set and the set T is restricted to be an interval unbounded from below, we obtain the class of *threshold graphs* [5], which is very well understood and admits several characterizations, as well as linear time algorithms for recognition and for several optimization problems [16]. If P denotes the property of being a dominating set and T is an interval unbounded from above, we obtain the class of *domishold graphs* [1], which enjoy similar properties as threshold graphs. On the other hand, if P is the property of being a *maximal* stable set and T is restricted to consist of a single number, we obtain the class of *equistable graphs* [18], for which the recognition complexity is open (see, e.g., [15]), no structural characterization is known, and the maximum size of a stable set in an equistable graph is hard to approximate [17].

As the above examples show, the resulting class of graphs can be either *hereditary* (that is, closed under vertex deletion)—as in the case of threshold or domishold graphs—, or non-hereditary—as in the case of equistable graphs. When the resulting graph class is not hereditary, it is natural to consider the hereditary version of the property, in which the requirement (the existence of weights and the set T) is extended to all induced subgraphs of the given graph.

In this paper, we introduce and study the case when P is the property of being a total dominating set and T is an interval unbounded from above. Given a graph $G = (V, E)$, a *total dominating set* (a TD set, for short) is a subset S of the vertices of G such that every vertex of G has a neighbor in S . For surveys of the literature on the subject of total domination, see [11–13].

Definition 1. A graph $G = (V, E)$ is said to be *total domishold* (TD for short) if there exists a pair (w, t) where $w : V \rightarrow \mathbb{R}_+$ is a weight function and $t \in \mathbb{R}_+$ is a threshold such that for every subset $S \subseteq V$, $w(S) := \sum_{x \in S} w(x) \geq t$ if and only if S is a total dominating set in G . A pair (w, t) as above will be referred to as a *total domishold structure* of G .

Remark 1. For convenience, the above definition allows G to have isolated vertices. Every graph with an isolated vertex is total domishold, even though it does not have any TD sets.

Example. The complete graph of order n is total domishold. Indeed, a subset $S \subseteq V(K_n)$ is a total dominating set of K_n if and only if S is of size at least two, and consequently the pair $(w, 2)$ where $w(x) = 1$ for all $x \in V(K_n)$ is a total domishold structure of K_n . On the other hand, the 4-cycle C_4 is not a total domishold graph (cf. Proposition 5 in Section 4).

It is easy to see that adding a new vertex to the 4-cycle and connecting it to exactly one vertex of the cycle results in a total domishold graph. Therefore, contrary to the classes of threshold and domishold graphs, the class of total domishold graphs is not hereditary. This motivates the following definition:

Definition 2. A graph G is said to be hereditary total domishold (*HTD for short*) if every induced subgraph of it is total domishold.

Our results. We initiate the study of TD and HTD graphs. We identify several operations preserving the class of TD graphs, which, together with results from the literature [1, 5], imply that the class of HTD graphs properly contain the classes of threshold graphs and the complements of domishold graphs (Section 3). As our main results, we obtain the following partial results towards a characterization of HTD graphs (in Section 4):

- (1) We identify a set \mathcal{F} of 13 forbidden induced subgraphs for the class of HTD graphs, which implies that every HTD graph is a $(1, 2)$ -polar chordal graph.
- (2) Split graphs form a well known class of $(1, 2)$ -polar chordal graphs. We characterize the HTD split graphs. The characterization is obtained by studying a new family of hypergraphs, defined similarly as the Sperner hypergraphs, which might be of independent interest.
- (3) An *extension* of a graph G is the graph obtained from G by attaching an arbitrary (non-negative) number of pendant vertices to each vertex of G . We characterize HTD extensions of threshold graphs. As a corollary of this characterization, we obtain a family of HTD graphs defined by a set \mathcal{F}' of 13 forbidden induced subgraphs closely related to the set \mathcal{F} mentioned above.

We conclude in Section 5 by showing that TD graphs can be recognized in polynomial time, and developing a polynomial time algorithm to find a minimum total dominating set in a given TD graph.

2 Preliminaries and Notation

Graphs and graph classes. A graph G is *chordal* if it does not contain any induced cycle of order at least 4, *split* if its vertex set can be partitioned into a clique and an independent set, and $(1, 2)$ -*polar* if it admits a partition of its vertex set into two (possibly empty) parts K and L such that K is a clique and L induces a subgraph of maximum degree at most 1. For a set of graphs \mathcal{F} , a graph G is said to be \mathcal{F} -*free* (or just F -*free* if $\mathcal{F} = \{F\}$), if it does not contain any induced subgraph isomorphic to a member of \mathcal{F} . Every member of \mathcal{F} is said to be a *forbidden induced subgraph* for the (hereditary) set of \mathcal{F} -free graphs. The neighborhood of a vertex v in a graph will be denoted by $N_G(v)$, and its closed neighborhood by $N_G[v] := N_G(v) \cup \{v\}$, omitting the subscript G if the graph is clear from the context. A vertex in a graph G is *universal* if it is adjacent to every other vertex in G and *isolated* if it is of degree 0. By $G + H$ we will denote the disjoint union of graphs G and H . The *join* of graphs G and H is the graph obtained from the disjoint union $G + H$ by adding all edges of the form $\{uv \mid u \in V(G), v \in V(H)\}$. For a graph G , we denote by $2G$ the disjoint

union of two copies of G . As usual, we denote by K_n , P_n and C_n the complete graph, the path and the cycle on n vertices.

The following characterization of threshold graphs due to Chvátal and Hammer will be used in some of our proofs.

Theorem 1 (Chvátal and Hammer [5]). *For every threshold graph G the following conditions are equivalent:*

- a) G is threshold.
- b) G has no induced subgraph isomorphic to $2K_2$, C_4 or P_4 .
- c) There is a partition of V into two disjoint sets I, K and an ordering u_1, u_2, \dots, u_k of vertices in I such that no two vertices in I are adjacent, every two vertices in K are adjacent, and $N(u_1) \subseteq N(u_2) \subseteq \dots \subseteq N(u_k)$. Without loss of generality, we will assume that I is a maximal independent set of G , that is, that $\cup_{u \in I} N(u) = K$.

Boolean functions. A Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is *positive* if $f(x) \leq f(y)$ holds for every two vectors $x, y \in \{0, 1\}^n$ such that $x \leq y$ (that is, $x_i \leq y_i$ for all $i \in \{1, \dots, n\}$). A positive Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ is *threshold* if there exist non-negative real weights $w = (w_1, \dots, w_n)$ and a non-negative real number t such that for every $x \in \{0, 1\}^n$, $f(x) = 0$ if and only if $\sum_{i=1}^n w_i x_i \leq t$ (see, e.g., [7]). Such a pair (w, t) is called a *separating structure* of f . Every threshold positive Boolean function admits an integral separating structure [7]. The problem of determining whether a positive Boolean function given by its complete DNF is threshold is solvable in polynomial time, using dualization and linear programming. This result is summarized in the following theorem.

Theorem 2 (Peled and Simeone [19], see also Theorem 9.16 in [7]). *There exists a polynomial time algorithm that determines, given the complete DNF of a positive Boolean function $f(x_1, \dots, x_n)$, whether f is threshold. If this is the case, the algorithm also computes an integral separating structure of f .*

Hypergraphs. A *hypergraph* is a pair $H = (V, E)$ where V is a finite set of *vertices* and E is a set of subsets of V , called (*hyper*)*edges* [2]. A hypergraph $H = (V, E)$ is *threshold* if there exist a weight function $w : V \rightarrow \mathbb{R}_+$ and a threshold $t \in \mathbb{R}_+$ such that for all subsets $X \subseteq V$, it holds $w(X) \leq t$ if and only if X contains no edge of H [10]. Reformulating the characterization of threshold positive Boolean functions due to Chow [4] and Elgot [8] (see also Theorem 9.14 in [7]) in the language of hypergraphs, we obtain the following characterization of threshold hypergraphs.

Theorem 3 (Chow [4], Elgot [8]). *A hypergraph $H = (V, E)$ is not threshold if and only if there exists an integer n with $n \geq 2$ and n (not necessarily distinct) subsets A_1, \dots, A_n of V , each containing an edge of H , and n (not necessarily distinct) subsets B_1, \dots, B_n of V , each containing no edge of H , such that for every vertex $v \in V$,*

$$|\{i \mid v \in A_i\}| = |\{i \mid v \in B_i\}|. \quad (1)$$

For a positive integer n , we will use the notation $[n]$ for the set $\{1, \dots, n\}$.

3 Basic Properties of TD Graphs

In this section, we establish some basic properties of TD graphs.

Proposition 1. *Every graph with an isolated vertex is TD.*

Proof. If G has an isolated vertex, then G does not have any TD sets, and hence the pair $(w, |V(G)| + 1)$, where $w(x) = 1$ for all $x \in V(G)$ is a total domishold structure of G . \square

As shown by the 4-cycle, TD graphs are not closed under join. However, they are closed under join with K_1 , that is, under adding a universal vertex. This is stated formally in Proposition 2 and proved using the following auxiliary lemma.

Lemma 1. *Every TD graph admits a total domishold structure in which all weights are positive.*

Proof. Let (w, t) be a total domishold structure of a TD graph $G = (V, E)$. The value of

$$\delta = t - \max\{w(S) \mid S \in \mathcal{P}(V) \setminus \mathcal{T}\},$$

where $\mathcal{P}(V)$ denotes the power set of V and \mathcal{T} denotes the set of all total dominating sets of G , is well defined and positive. Let $w' : V \rightarrow \mathbb{R}_+ \setminus \{0\}$ and $t' \in \mathbb{R}$ be defined as: $w'(x) = |V|w(x) + \delta/2$ for all $x \in V$, and $t' = |V|t$. We claim that (w', t') is a total domishold structure of G . On the one hand, if $S \in \mathcal{T}$, then $w'(S) = |V|w(S) + \delta|S|/2 \geq |V|t = t'$. On the other hand, if $S \in \mathcal{P}(V) \setminus \mathcal{T}$, then $w(S) + \delta/2 < t$ and consequently $w'(S) = |V|w(S) + \delta|S|/2 \leq |V|(w(S) + \delta/2) < |V|t = t'$. \square

Proposition 2. *Let G be a graph, and let G' be the graph obtained from G by adding to it a vertex adjacent to all vertices of G . Then, G is TD if and only if G' is TD.*

Proof. The proof will follow from the observation that the sets \mathcal{T} and \mathcal{T}' of total dominating sets of G and G' , respectively, are related as follows:

$$\mathcal{T}' = \mathcal{T} \cup \{\{v\} \cup S \mid \emptyset \neq S \subseteq V(G)\},$$

where v is the added vertex.

Suppose first that G is TD. By Lemma 1, G admits a total domishold structure (w, t) with $w(x) > 0$ for all $x \in V(G)$. Let $w' : V(G') \rightarrow \mathbb{R}_+$ be defined as follows: for all $x \in V(G)$, let $w'(x) = w(x)$; let $w'(v) = t - \min\{w(x) \mid x \in V(G)\}$. We claim that (w', t) is a total domishold structure of G' . Indeed, if $S \in \mathcal{T}'$ then we consider two cases. If $v \notin S$, then $S \in \mathcal{T}$ and $w'(S) = w(S) \geq t$. If $v \in S$, then $\{x, v\} \subseteq S$ for some $x \in V(G)$, and hence $w'(S) \geq w'(x) + w'(v) = w(x) + t - \min\{w(z) \mid z \in V(G)\} \geq t$. Similarly, if $w'(S) \geq t$, we consider two cases. If $v \notin S$, then $w(S) \geq t$ and therefore $S \in \mathcal{T} \subseteq \mathcal{T}'$. If $v \in S$, then $S \cap V(G) \neq \emptyset$ (since otherwise we would have $w'(S) = w'(v) < t$ by the positivity of w), and thus $S \in \mathcal{T}'$.

The other direction is straightforward. Since $\mathcal{T}' \cap \mathcal{P}(V(G)) = \mathcal{T}$ (where $\mathcal{P}(\cdot)$ is the power set operator), any pair (w, t) such that (w', t) is a total domishold structure of G' and w is the restriction of w' to $V(G)$, is a total domishold structure of G . \square

Corollary 1. *Every threshold graph is HTD.*

Proof. Chvátal and Hammer proved in [5] that the class of threshold graphs is hereditary, and that every threshold graph contains either an isolated vertex or a universal vertex. Therefore, an induction on the number of vertices together with Propositions 1 and 2 shows that every threshold graph is TD. Since the class of threshold graphs is hereditary, every threshold graph is HTD. \square

In general, TD graphs are not closed under disjoint union: the path P_3 is TD, but the graph $2P_3$ is not (cf. Proposition 5 in Section 4). However, they are closed under adding a (TD) graph with a *unique* (inclusion-wise) minimal TD set.

Proposition 3. *Let G and H be graphs such that H has a unique minimal TD set. Then, $G + H$ is TD if and only if G is TD.*

Proof. Let T be the unique minimal TD set in H . Then, the sets \mathcal{T} and \mathcal{T}' of total dominating sets of G and $G' := G + H$ are related as follows:

$$\mathcal{T}' = \{S \cup T' \mid S \in \mathcal{T} \text{ and } T \subseteq T' \subseteq V(H)\}.$$

Suppose that G is TD, with a total domishold structure (w, t) . Let $N = w(V(G))$ and define $w' : V(G') \rightarrow \mathbb{R}_+$ as

$$w'(x) = \begin{cases} w(x), & \text{if } x \in V(G); \\ N, & \text{if } x \in T; \\ 0, & \text{otherwise.} \end{cases}$$

It is not hard to verify that the pair $(w', t + |T|N)$ is a total domishold structure of G' .

Conversely, if (w', t') is a total domishold structure of G' , then $(w, t' - w'(T))$, where w is the restriction of w' to $V(G)$, is a total domishold structure of G . \square

Corollary 2. *Let G be a graph, and let $G' = G + K_2$. Then, G is TD if and only if G' is TD.*

A graph G is said to be *co-domishold* if its complement is domishold. Since threshold graphs are exactly the domishold co-domishold graphs [1, 5], the following result generalizes Corollary 1.

Corollary 3. *Every co-domishold graph is HTD.*

Proof. This can be proved similarly as Corollary 1, using Corollary 2 in addition to Propositions 1 and 2, and the facts that: (1) the class of co-domishold graphs is hereditary (this is because the class of domishold graph is hereditary [1]); (2) every co-domishold graph contains either an isolated vertex, a universal vertex, or a connected component isomorphic to K_2 [1]. \square

As observed in the introduction, the set of TD graphs is not hereditary. We now strengthen this observation by showing that the set of TD graphs is not contained in any nontrivial hereditary class of graphs (even if we disallow graphs with isolated vertices).

Proposition 4. *For every graph G there exists a TD graph G' without isolated vertices such that G is an induced subgraph of G' .*

Proof. Let G be a graph. First, add to $G = (V, E)$ a new vertex, say v , and connect v only to isolated vertices of G . Second, add a new private neighbor to every vertex of the resulting graph. Denoting by G' the obtained graph, it is clear that G is an induced subgraph of G' . By construction, the set $V \cup \{v\}$ is the unique minimal total dominating set in G' . Therefore, the pair (w, t) , where $w : V(G') \rightarrow \mathbb{R}_+$ is given by $w(x) = 1$ if $x \in V \cup \{v\}$ and $w(x) = 0$, otherwise, and $t = |V| + 1$, is a total domishold structure of G' . \square

We conclude this section with a characterization of TD graphs in terms of the thresholdness of a derived Boolean function, a characterization that will turn out useful in proofs in later sections.

3.1 A Characterization of TD Graphs in Boolean Terms

We first fix some terminology and notations. Given a set V and a binary vector $x \in \{0, 1\}^V$, the *support set* of a vector $x \in \{0, 1\}^V$ is the set $S(x) = \{v \in V \mid x_v = 1\}$. Also, by \bar{x} we denote the vector $\bar{x} \in \{0, 1\}^V$ given by $(\bar{x})_i = 1 - x_i$ for all $i \in V$.

Definition 3. *Given a graph $G = (V, E)$, its neighborhood function is the positive Boolean function $f_G : \{0, 1\}^V \rightarrow \{0, 1\}$ that takes value 1 precisely on vectors $x \in \{0, 1\}^V$ whose support set $S(x)$ contains the neighborhood of some vertex of G . Formally,*

$$f_G(x) = \bigvee_{v \in V} \bigwedge_{u \in N(v)} x_u$$

for every vector $x \in \{0, 1\}^V$. (If $N(v) = \emptyset$ then $\bigwedge_{u \in N(v)} x_u = 1$.)

Lemma 2. *A graph $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ is total domishold if and only if its neighborhood function f_G is threshold. Moreover, if (w_1, \dots, w_n, t) is an integral separating structure of f_G , then $(w; \sum_{i=1}^n w_i - t)$ with $w(v_i) = w_i$ for all $i \in [n]$ is a total domishold structure of G .*

Proof. First, recall that a positive Boolean function $f(x_1, \dots, x_n)$ is threshold if and only if its dual function $f^d(x) = \overline{f(\bar{x})}$ is threshold, and that if (w_1, \dots, w_n, t) is an integral separating structure of f , then $(w_1, \dots, w_n, \sum_{i=1}^n w_i - t - 1)$ is a separating structure of f^d [7]. Therefore, it suffices to argue that G is total domishold if and only if f_G^d is threshold.

Let $x \in \{0, 1\}^V$ and let $S(x)$ be the support set of x . By definition, $f_G^d(x) = 0$ if and only if $f(\bar{x}) = 1$, which is the case if and only if $V \setminus S$ contains the neighborhood of some vertex. In other words, $f_G^d(x) = 0$ if and only if S is not a total dominating set. Hence, if the dual function f_G^d is threshold with an integral separating structure (w_1, \dots, w_n, t) , then $(w, t + 1)$ with $w(v_i) = w_i$ for all $i \in [n]$ is a total domishold structure of G , and conversely, if (w, t) is an integral total domishold structure of G , then $(w_1, \dots, w_n, t - 1)$ with $w_i = w(v_i)$ for all $i \in [n]$ is a separating structure of f_G^d .

Finally, if (w_1, \dots, w_n, t) is an integral separating structure of f_G , then $(w_1, \dots, w_n, \sum_{i=1}^n w_i - t - 1)$ is a separating structure of f_G^d and hence $(w; \sum_{i=1}^n w_i - t)$ with $w(v_i) = w_i$ for all $i \in [n]$ is a total domishold structure of G . \square

4 Partial Characterizations of HTD Graphs

In this section, we obtain some partial results towards a characterization of hereditary total domishold graphs. We start by identifying 13 forbidden induced subgraphs for the set of HTD graphs.

Proposition 5. *Every HTD graph is $\{F_1, \dots, F_{13}\}$ -free, where F_1, \dots, F_{13} are the graphs depicted in Fig. 1.*

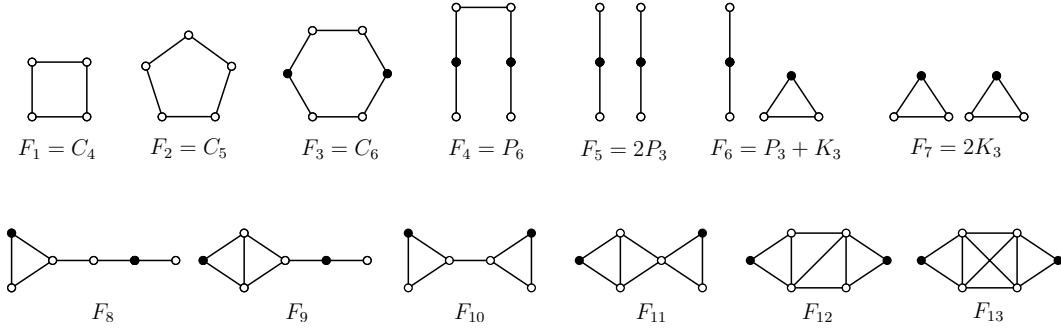


Figure 1: Graphs F_1, \dots, F_{13} .

Proof. We only need to verify that none of the graphs F_1, \dots, F_{13} is TD. We will verify this independently first for $F_1 = C_4$ and $F_2 = C_5$, and then for the remaining 11 graphs with a single argument. Let $F \in \{F_1, \dots, F_{13}\}$.

Case 1. $F = F_1 = C_4$.

Suppose for a contradiction that F , with vertices v_1, \dots, v_4 in circular order, admits a total domishold structure (w, t) . Since $\{v_1, v_2\}$ is a TD set of F , we have $w(v_1) + w(v_2) \geq t$. Similarly, $w(v_3) + w(v_4) \geq t$. Therefore, $w(V(F)) = w(v_1) + w(v_2) + w(v_3) + w(v_4) \geq 2t$. On the other hand, since neither of the sets $\{v_1, v_3\}$ and $\{v_2, v_4\}$ is a TD set of F , it holds that $w(v_1) + w(v_3) < t$ and $w(v_2) + w(v_4) < t$, implying $w(V(F)) < 2t$. This contradiction shows that C_4 is not a TD graph.

Case 2. $F = F_2 = C_5$.

Suppose that F , with vertices v_1, \dots, v_5 in circular order, admits a total domishold structure (w, t) .

For every $i \in [5]$, consider the set $D_i = \{v_{i-1}, v_i, v_{i+1}\}$ (indices modulo 5). Then, every D_i is a total dominating set of F . Moreover, every vertex $v_i \in V(F)$ is contained in precisely 3 such sets D_j . Consequently, we have

$$3w(V(F)) = \sum_{v \in V(F)} \sum_{\substack{1 \leq i \leq 5 \\ v \in D_i}} w(v) = \sum_{i=1}^5 \sum_{v \in D_i} w(v) = \sum_{i=1}^5 w(D_i) \geq 5t. \quad (2)$$

On the other hand, for every $i \in [5]$, the set $S_i := \{v_{i-2}, v_i, v_{i+2}\}$ is not a total dominating set of F (since v_i has no neighbor in S_i), and every vertex $v_i \in V(F)$ is contained in precisely

3 such sets S_j . Therefore

$$3w(V(F)) = \sum_{v \in V(F)} \sum_{\substack{1 \leq i \leq 5 \\ v \in S_i}} w(v) = \sum_{i=1}^5 \sum_{v \in S_i} w(v) = \sum_{i=1}^5 w(S_i) < 5t,$$

a contradiction with (2). Hence $F_2 = C_5$ is not a TD graph.

Case 3. $F = F_i$ for some $i \in \{3, 4, \dots, 13\}$. Then, F is a graph with 6 vertices such that there exists a pair a, b of vertices of degree 2 in F with pairwise disjoint closed neighborhoods. In the drawing of F in Fig. 1, such a pair consists of the two black vertices.

Suppose that F admits a total domishold structure (w, t) . Let $N(a) = \{a_1, a_2\}$ and $N(b) = \{b_1, b_2\}$. Since every vertex is contained in the closed neighborhood of either a or b , we have $V(F) = \{a, a_1, a_2, b, b_1, b_2\}$. Moreover, the set $D_1 := \{a, a_1, b, b_1\}$ is a total dominating set of F , and so is the set $D_2 := \{a, a_2, b, b_2\}$. In particular, since (w, t) is a total domishold structure of F , it holds that $w(D_1) \geq t$ and $w(D_2) \geq t$. Consequently,

$$w(a) + w(b) + w(V(F)) = w(D_1) + w(D_2) \geq 2t. \quad (3)$$

On the other hand, the set $S_a := \{a, b, b_1, b_2\}$ is not a total dominating set of F , since a has no neighbors in S . Therefore, $w(S_a) < t$. Similarly, the set $S_b := \{b, a, a_1, a_2\}$ is not a total dominating set of F , and $w(S_b) < t$. We thus obtain

$$w(a) + w(b) + w(V(F)) = w(S_a) + w(S_b) < 2t,$$

contrary to (3). This contradiction shows that F is not a TD graph, and completes the proof of the proposition. \square

Proposition 5 implies a nice structural feature of HTD graphs. Recall that a graph is said to be $(1, 2)$ -polar if it admits a partition of its vertex set into two (possibly empty) parts K and L , such that K is a clique and L induces a subgraph of maximum degree at most 1.

Corollary 4. *Every HTD graph is a $(1, 2)$ -polar chordal graph.*

Proof. In [9], Gagarin and Metelskii characterized the set of $(1, 2)$ -polar graphs with a set of 18 forbidden induced subgraphs (see also [20]). Only three graphs in this list are chordal: $2P_3$, $P_3 + K_3$, and $2K_3$ (that is, the graphs F_5, F_6 , and F_7 from Fig. 1). This implies that the class of $(1, 2)$ -polar chordal graphs is exactly the class of $\{F_5, F_6, F_7, C_4, C_5, C_6, \dots\}$ -free graphs. Notice that $C_4 = F_1$, $C_5 = F_2$, $C_6 = F_3$, and $F_4 = P_6$ is an induced subgraph of every cycle of order at least 7. Therefore, the class of $\{F_1, \dots, F_7\}$ -free graphs is a subclass of the class of $(1, 2)$ -free polar chordal graphs. In particular, by Proposition 5, the same is true for HTD graphs. \square

Notice that the converse of Corollary 4 does not hold: graphs F_8, F_9, \dots, F_{13} are $(1, 2)$ -polar chordal graphs that are not TD.

In the rest of this section, we give a complete characterization of HTD graphs within two subclasses of $(1, 2)$ -polar chordal graphs: the class of split graphs, and the class of the extensions of threshold graphs.

4.1 Split HTD Graphs

Recall that a graph G is split if its vertex set can be partitioned into a clique and an independent set. Our characterization of split HTD graphs will be based on a new family of hypergraphs, defined similarly as the Sperner hypergraphs. A hypergraph $H = (V, E)$ is said to be *Sperner* (or: a *clutter*) if no edge of H contains another edge, or, equivalently, if for every two distinct edges e and f of H , it holds that $\min\{|e \setminus f|, |f \setminus e|\} \geq 1$. This motivates the following definition.

Definition 4. A hypergraph $H = (V, E)$ is said to be dually Sperner if for every two distinct edges e and f of H , it holds that

$$\min\{|e \setminus f|, |f \setminus e|\} \leq 1.$$

Lemma 3. Every dually Sperner hypergraph is threshold.

Proof. Suppose for a contradiction that there exists a dually Sperner hypergraph $H = (V, E)$ that is not threshold. By Theorem 3, there exists an integer $n \geq 2$ and n (not necessarily distinct) subsets A_1, \dots, A_n of V , each containing an edge of H , and n (not necessarily distinct) subsets B_1, \dots, B_n of V , each containing no edge of H , such that for every vertex $v \in V$, condition (1) holds. For every $i \in [n]$, let e_i be an edge of H contained in A_i . Let $i^* \in [n]$ be such that $|e_{i^*}| \leq |e_i|$ for all $i \in [n]$. In particular, this implies that for every $i \in [n]$, it holds that $|e_{i^*} \setminus e_i| \leq |e_i \setminus e_{i^*}|$, which, since H is dually Sperner, implies

$$|e_{i^*} \setminus e_i| \leq 1 \tag{4}$$

for every $i \in [n]$. On the other hand, since no B_i contains the edge e_{i^*} , we have, for all $i \in [n]$, the inequality

$$1 \leq |e_{i^*} \setminus B_i|. \tag{5}$$

Adding up the inequalities (5) for all $i \in [n]$, we obtain

$$n \leq \sum_{i \in [n]} |e_{i^*} \setminus B_i|.$$

This implies the following contradicting chain of equations and inequalities

$$\begin{aligned} n &\leq \sum_{i \in [n]} |e_{i^*} \setminus B_i| = \sum_{i \in [n]} \sum_{v \in e_{i^*} \setminus B_i} 1 = \sum_{v \in e_{i^*}} \sum_{i: v \notin B_i} 1 = \sum_{v \in e_{i^*}} (n - |\{i : v \in B_i\}|) \\ &= \sum_{v \in e_{i^*}} (n - |\{i : v \in A_i\}|) = \sum_{v \in e_{i^*}} \sum_{i: v \notin A_i} 1 = \sum_{i \in [n]} \sum_{v \in e_{i^*} \setminus A_i} 1 = \sum_{i \in [n]} |e_{i^*} \setminus A_i| \\ &\leq \sum_{i \in [n]} |e_{i^*} \setminus e_i| = \sum_{\substack{i \in [n] \\ i \neq i^*}} |e_{i^*} \setminus e_i| \leq \sum_{\substack{i \in [n] \\ i \neq i^*}} 1 = n - 1. \end{aligned}$$

The first equality in the second line follows from condition (1), while the first inequality in the third line follows from the fact that $e_i \subseteq A_i$, which implies $e_{i^*} \setminus A_i \subseteq e_{i^*} \setminus e_i$. The last inequality follows from (4).

This contradiction completes the proof. \square

Using Lemma 3, we can now derive the following characterization of split HTD graphs.

Theorem 4. *Let $G = (V, E)$ be a split graph. Then, the following statements are equivalent:*

1. *G is hereditary total domishold.*
2. *G is F_{13} -free (see Fig. 1).*

Proof. The implication (1) \Rightarrow (2) follows immediately from Proposition 5.

For the implication (2) \Rightarrow (1), let $G = (V, E)$ be an F_{13} -free split graph. Since the class of F_{13} -free split graphs is hereditary, it is enough to show that G is total domishold. We prove this by induction on $|V|$. For $|V| = 1$, the graph G is K_1 and hence TD. Let $|V| > 1$. By Proposition 2 and the inductive hypothesis, we may assume that G has no universal vertices. Let $V = K \cup I$ where K is a clique, I is an independent set, and $K \cap I = \emptyset$. By Lemma 2, it suffices to show that the neighborhood function $f_G(x) = \bigvee_{v \in V} \bigwedge_{u \in N(v)} x_u$ is threshold. Notice that since G has no universal vertices, for every vertex $v \in K$ there exists a vertex $u \in I$ such that $N(u) \subseteq N(v)$. In particular, this implies that the neighborhood function of G is logically equivalent to the function $g : \{0, 1\}^V \rightarrow \{0, 1\}$ given by $g(x) = \bigvee_{v \in I} \bigwedge_{u \in N(v)} x_u$. Consider the hypergraph $H = (K, \{N(v) \mid v \in I\})$. Since G is F_{13} -free, H is dually Sperner, and by Lemma 3, H is threshold. Therefore, there exist a weight function $w : K \rightarrow \mathbb{R}_+$ and a threshold $t \in \mathbb{R}_+$ such that for all subsets $X \subseteq K$, it holds $w(X) \leq t$ if and only if X contains no neighborhood of a vertex in I . Let $w' : V \rightarrow \mathbb{R}_+$ be the extension of w that agrees with w on K and assigns 0 to every vertex in I . The definition of g implies that for all $x \in \{0, 1\}^V$, we have $g(x) = 0$ if and only if the set $S(x) \cap K$, where $S(x)$ is the support set of K , contains no neighborhood of a vertex in I . Consequently, the pair (w', t) is a separating structure of g , which implies that g is threshold, and so is f_G . By Lemma 2, G is total domishold. \square

4.2 HTD Extensions of Threshold Graphs

An *extension* of a graph G is the graph obtained from G by attaching an arbitrary (non-negative) number of pendant vertices to each vertex of G . In this section, we obtain a characterization of HTD extensions of threshold graphs. As a corollary of this characterization, we will show that replacing F_{11} and F_{12} in Fig. 1 with the graphs F'_{11} and F'_{12} obtained from F_{11} and F_{12} respectively by deleting a particular vertex, we obtain a set \mathcal{F}' of graphs such that every \mathcal{F}' -free graph is HTD.

Definition 5. *Given a graph G and a function $\ell : V(G) \rightarrow \mathbb{Z}_+$, the ℓ -extension of G is the graph G_ℓ^+ obtained from G by appending to every vertex v of G exactly $\ell(v)$ leaves. More formally:*

- $V(G_\ell^+) = V(G) \cup L$, where $L = \bigcup_{v \in V(G)} L_v$ such that $L \cap V(G) = \emptyset$, $|L_v| = \ell(v)$ for all $v \in V(G)$, and $L_u \cap L_v = \emptyset$ for distinct $u, v \in V(G)$.
- $E(G_\ell^+) = E(G) \cup \bigcup_{v \in V(G)} \{vv' \mid v' \in L_v\}$.

The following theorem, the proof of which is deferred to Section 4.3, characterizes HTD extensions of threshold graphs.

Theorem 5. *Let G' be an extension of a threshold graph G . Then, the following are equivalent:*

1. G' is HTD.
2. G' is $\{2P_3, P_3 + K_3\}$ -free.
3. *If $\ell : V(G) \rightarrow \mathbb{Z}_+$ is the function such that G' is the ℓ -extension of G , then the following conditions are satisfied:*
 - a) *For every two distinct non-adjacent vertices $u, v \in V(G)$, it holds that $\ell(u) \leq 1$ or $\ell(v) \leq 1$.*
 - b) *For every vertex $u \in V(G)$ and every pair of adjacent non-neighbors x, y of u , either $\ell(u) \leq 1$ or $\ell(x) = \ell(y) = 0$.*
 - c) *For every vertex $u \in V(G)$ such that there exists a (not necessarily induced) P_3 in $G - N_G[u]$, it holds that $\ell(u) \leq 1$.*

The link between Theorem 5 and the result that every \mathcal{F}' -free graph is HTD is provided by the following lemma.

Lemma 4. *Let G be a connected $\{F_1, F_2, F_3, F_4, F_8, F_9, F_{10}, F_{11}^-, F_{12}^-, F_{13}\}$ -free graph (see Fig. 2). Then, G is an extension of a threshold graph.*

Proof. Suppose for a contradiction that there exists a connected \mathcal{F} -free graph G with $\mathcal{F} = \{F_1, F_2, F_3, F_4, F_8, F_9, F_{10}, F_{11}^-, F_{12}^-, F_{13}\}$ that is not an extension of a threshold graph. In particular, the subgraph H of G induced by the set of vertices of degree at least 2 is not threshold. By Theorem 1, H contains an induced subgraph isomorphic to $2K_2, C_4$ or P_4 . Since G is C_4 -free, so is H . Therefore, H contains an induced subgraph isomorphic to $2K_2$ or P_4 . We consider two cases, depending on whether H has an induced subgraph isomorphic to $2K_2$ or not.

Case 1: H has an induced subgraph isomorphic to $2K_2$.

Let \mathcal{K} denote the set of all induced subgraph of H isomorphic to $2K_2$. Since G is connected, the two edges of every $K \in \mathcal{K}$ are connected by a path in G . Let $d(K)$ denote the distance in G between the two edges of K . Choose a $K \in \mathcal{K}$ that minimizes the value of $d(K)$. Let us write $V(K) = \{a, b, c, d\}$ so that $E(K) = \{ab, cd\}$. Let $P = (x_0, x_1, \dots, x_{k-1}, x_k)$ be a shortest $\{a, b\} - \{c, d\}$ path in G . Then, $x_0 \in \{a, b\}$ and $x_k \in \{c, d\}$. Since K is an induced subgraph of G , we have $k \geq 2$. Moreover, $k = 2$ since if $k \geq 3$, the set $\{a, b, x_2, x_3\}$ would induce a subgraph K' of H isomorphic to $2K_2$, with $d(K') < d(K)$, contrary to the choice of K .

Without loss of generality, we may assume that $x_0 = a$ and $x_2 = c$. For simplicity, let us also write $x = x_1$. Since G is F_{11}^- -free, we may assume without loss of generality that x is not adjacent to d .

We consider two further subcases, depending on whether x is adjacent to b or not.

Case 1.1: x is adjacent to b .

Since $d \in V(H)$, the degree of d in G is at least 2. Thus, vertex d must have a neighbor in G in the set $V(G) \setminus \{a, b, c, x\}$, say y .

If y is not adjacent to c , then using the C_4 -freeness of G we infer that y is also not adjacent to x , and moreover, due to the C_5 -freeness of G , y is also not adjacent to either

of b and c . But now, an induced subgraph of G isomorphic to F_8 arises on the vertex set $\{a, b, c, d, x, y\}$, a contradiction. Therefore, y is adjacent to c .

If y is adjacent to x , then using the F_{12}^- -freeness of G we infer that y is not adjacent to either of b and c . But now, an induced subgraph of G isomorphic to F_{11}^- arises on the vertex set $\{a, b, c, x, y\}$, a contradiction. Therefore, y is not adjacent to x .

The C_4 -freeness of G implies that y is not adjacent to either of b and c . But now, an induced subgraph of G isomorphic to F_{10} arises on the vertex set $\{a, b, c, d, x, y\}$, a contradiction.

Case 1.2: x is not adjacent to b .

Since $b \in V(H)$, the degree of b in G is at least 2. Thus, vertex b must have a neighbor in G in the set $V(G) \setminus \{a, c, d, x\}$, say y .

If y is not adjacent to a , then, using the C_4 -freeness of G we infer that y is also not adjacent to x . Due to the C_5 -freeness of G , we infer that y is also not adjacent to c , and, due to the C_6 -freeness of G , also not to d . But now, an induced subgraph of G isomorphic to P_6 arises on the vertex set $\{a, b, c, d, x, y\}$, a contradiction. Therefore, y is adjacent to a .

If y is not adjacent to x , then the C_4 -freeness of G implies that y is not adjacent to c , and the C_5 -freeness of G implies that y is not adjacent to d . But now, an induced subgraph of G isomorphic to F_8 arises on the vertex set $\{a, b, c, d, x, y\}$, a contradiction. Therefore, y is adjacent to x .

The F_{12}^- -freeness of G implies that y is not adjacent to c , and the C_4 -freeness of G implies that y is not adjacent to d . But now, an induced subgraph of G isomorphic to F_9 arises on the vertex set $\{a, b, c, d, x, y\}$, a contradiction.

Case 2: H has no induced subgraph isomorphic to $2K_2$.

In this case, H has an induced subgraph K isomorphic to P_4 . Let us write $V(K) = \{a, b, c, d\}$ so that $E(K) = \{ab, bc, cd\}$. Since $a, d \in V(H)$, the degrees of a and d in G are at least 2. Let us denote by x and y two respective neighbors of a and d in G in the set $V(G) \setminus \{a, b, c, d\}$.

If x is adjacent to d , then the C_5 -freeness of G implies that x is adjacent to either b or c . But now, G contains a subgraph isomorphic to either F_{12}^- (if x is adjacent to both b and c) or to C_4 (otherwise). In either case, we get a contradiction, and hence x is not adjacent to d . Similarly, y is not adjacent to a . In particular, this implies that $x \neq y$.

Suppose that x and y are adjacent. Then, x and y are both vertices of H , and to avoid an induced subgraph of H isomorphic to $2K_2$, we infer that x is adjacent to c , and similarly, that y is adjacent to b . Moreover, to avoid an induced subgraph of G isomorphic to C_4 , we infer that x is adjacent to b , and similarly, that y is adjacent to c . But now, an induced subgraph of G isomorphic to F_{13} arises on the vertex set $\{a, b, c, d, x, y\}$, a contradiction. Therefore, x and y are not adjacent.

To avoid an induced subgraph of G isomorphic to P_6 , we may assume without loss of generality that x is adjacent to either b or c . In particular, x belongs to $V(H)$. On the other hand, since the set $\{a, x, d, y\}$ induces a subgraph of G isomorphic to $2K_2$, we conclude that y does not belong to $V(H)$ (since otherwise H would contain an induced subgraph isomorphic to $2K_2$). In particular, y is a vertex of degree 1 in G , and hence y is not adjacent to b or c . Moreover, to avoid a subgraph of H isomorphic to $2K_2$ (induced on the vertex set $\{a, x, c, d\}$), we conclude that x is adjacent to c . The C_4 -freeness of G implies that x is also adjacent to b . But now, an induced subgraph of G isomorphic to F_9 arises on the vertex set $\{a, b, c, d, x, y\}$, a contradiction.

This completes the proof. \square

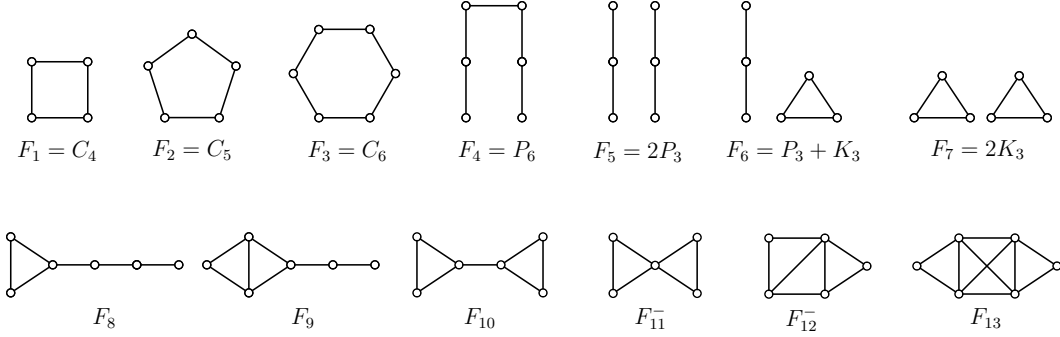


Figure 2: Graphs $F_1, \dots, F_{10}, F_{11}^-, F_{12}^-, F_{13}$.

Corollary 5. *Every $\{F_1, \dots, F_{10}, F_{11}^-, F_{12}^-, F_{13}\}$ -free graph is HTD, where $F_1, \dots, F_{10}, F_{11}^-, F_{12}^-, F_{13}$ are the graphs depicted in Fig. 2.*

Proof. Clearly, it is enough to prove that every $\{F_1, \dots, F_{10}, F_{11}^-, F_{12}^-, F_{13}\}$ -free graph G is total domishold. Since G is $\{2P_3, P_3 + K_3, 2K_3\}$ -free, G has at most one connected component with at least three vertices. Therefore, by Proposition 1 and Corollary 2 we may assume that G is a connected graph with at least three vertices. Lemma 4 implies that G is an extension of a threshold graph. Finally, since G is $\{2P_3, P_3 + K_3\}$ -free, Theorem 5 implies that G is TD (in fact, even HTD). \square

4.3 Proof of Theorem 5

The implication (1) \Rightarrow (2) follows immediately from Proposition 5.

(2) \Rightarrow (3): Suppose for a contradiction that $G' = G_\ell^+$ is $\{2P_3, P_3 + K_3\}$ -free but that the function $\ell : V(G) \rightarrow \mathbb{Z}_+$ violates one of the conditions 3a, 3b or 3c.

If ℓ violates 3a, then there exists a pair of non-adjacent vertices $u, v \in V(G)$ such that $\ell(u) \geq 2$ and $\ell(v) \geq 2$. Then, G' contains a $2P_3$ induced by $\{u, v\} \cup L'_u \cup L'_v$ where $L'_u \subseteq L_u$ and $L'_v \subseteq L_v$ with $|L'_u| = |L'_v| = 2$.

If ℓ violates 3b, then there exists a vertex $u \in V(G)$ and a pair of adjacent non-neighbors x, y of u such that $\ell(u) \geq 2$ and not both of $\ell(x)$ and $\ell(y)$ are 0. Without loss of generality, we may assume that $\ell(x) \geq 1$. Then, G' contains a $2P_3$ induced by $\{u, x, x', y\} \cup L'_u$ where $x' \in L_x$ and $L'_u \subseteq L_u$ such that $|L'_u| = 2$.

If ℓ violates 3c, then there exists a vertex $u \in V(G)$ with $\ell(u) \geq 2$ such that there exists a (not necessarily induced) P_3 in $G - N_G[u]$. Let xyz be a P_3 in $G - N_G[u]$. If x and z are non-adjacent in G , then G' contains a $2P_3$ induced by $\{u, x, y, z\} \cup L'_u$ where $x' \in L_x$ and $L'_u \subseteq L_u$ such that $|L'_u| = 2$. Similarly, if x and z are adjacent in G , then G' contains an induced $P_3 + K_3$.

In either case, we reach a contradiction with the $\{2P_3, P_3 + K_3\}$ -freeness of G' . This completes the proof of (2) \Rightarrow (3).

It remains to show the implication (3) \Rightarrow (1). Suppose, by contradiction, that there exists a pair (G, ℓ) satisfying conditions 3a–3c and such that the graph $G' = G_\ell^+$ is not HTD. Take a counterexample (G, ℓ) minimizing the number of vertices of G_ℓ^+ . Our proof will proceed in several steps, as a sequence of claims.

Claim 1. G_ℓ^+ is not TD.

Proof. Since G_ℓ^+ is not HTD, there exists an induced subgraph of G_ℓ^+ that is not TD. However, we will now show that every *proper* induced subgraph of G_ℓ^+ is TD, and the claim will follow.

Let H be a proper induced subgraph of G_ℓ^+ . Then, H can be written as the disjoint union of $(H')_{\ell'}^+$ and rK_1 for some $r \geq 0$, where H' is the subgraph of H induced by $V(H) \cap V(G)$ (in particular, H' is a threshold graph) and $\ell'(v) = |L_v \cap V(H)|$ for all $v \in V(H')$. Since H' is an induced subgraph of G , and $\ell'(v) \leq \ell(v)$ for all $v \in V(H')$, the pair (H', ℓ') satisfies the conditions 3a–3c. Hence, by the assumption that G_ℓ^+ is a minimal counterexample, graph $(H')_{\ell'}^+$ is HTD, and in particular TD. By Proposition 1, H is TD as well. \square

Let (I, K) be a partition of $V(G)$ given by Theorem 1. Let $I_0 = \{u \in V(G) \mid N_G(u) = \emptyset\}$ be the set of isolated vertices in G . By maximality of I , we see that I_0 is a subset of I .

Claim 2. $I_0 = \emptyset$.

Proof. Suppose that $I_0 \neq \emptyset$. By minimality of (G, ℓ) , the graph $G_\ell^+ - N_{G_\ell^+}[I_0]$ is TD. Proposition 1 and Corollary 2 imply that the graph G_ℓ^+ is TD whenever $\ell(u) \leq 1$ for all $u \in I_0$. Hence, $\ell(u) \geq 2$ for some $u \in I_0$, and condition 3c implies that $G - I_0$ is isomorphic to K_2 . Moreover, condition 3b implies that $\ell(x) = 0$ for each of the two vertices $x \in V(G \setminus I_0)$. By condition 3a, every connected component of G_ℓ^+ is either a K_1 , a K_2 , or a star $K_{1,t}$ with $t \geq 2$, and there is at most one component isomorphic to some $K_{1,t}$. Hence, G_ℓ^+ can be built from K_1 by adding to it isolated vertices, a universal vertex, and adding some isolated vertices and/or edges (in this order, where each operation is optional), and is therefore TD by Propositions 1, 2, and Corollary 2. This is in contradiction with Claim 1, thus completing the proof of Claim 2. \square

Consider the following equivalence relation on I :

$$(\forall x, y \in I) (x \sim y \Leftrightarrow (N_G(x) = N_G(y))) .$$

By property (c) of Theorem 1, the set of equivalence classes of \sim can be linearly ordered according to their increasing neighborhoods:

$$I/\sim = \{I_1, I_2, \dots, I_r\}$$

where $N_G(I_1) \subsetneq N_G(I_2) \subsetneq \dots \subsetneq N_G(I_r)$. Let also $K_i = N_G(I_i) \setminus N_G(I_{i-1})$ for all $i \in \{1, \dots, r\}$, and, for notational convenience, let $K_0 = \emptyset$.

Let $X = \{v \in V(G) \mid \ell(v) > 0\}$ be the set of vertices of G adjacent to a vertex in L . Notice that $X \subseteq D$ for every total dominating set D in G_ℓ^+ , since every vertex of X is the unique neighbor in G_ℓ^+ of some vertex in L .

Claim 3. $I_1 \subseteq X$.

Proof. Suppose for a contradiction that there exists a vertex $v \in I_1 \setminus X$. Then, every total dominating set of G_ℓ^+ must contain a vertex from $N_{G_\ell^+}(v) = N_G(v) = K_1$. We consider two cases, depending on whether X contains an element of K_1 or not.

Case 1: $X \cap K_1 \neq \emptyset$.

If $|X| \geq 2$, then X is the only minimal total dominating set (mtds for short) in G_ℓ^+ . Therefore, the pair (w, t) with

$$w(x) = \begin{cases} 1, & \text{if } x \in X; \\ 0, & \text{otherwise,} \end{cases}$$

and $t = |X|$ is a total domishold structure of G_ℓ^+ .

If $X = \{x\}$ for some $x \in K_1$, then every mtds of G_ℓ^+ consists of x and an arbitrary other vertex of G_ℓ^+ . Therefore, we can obtain a total domishold structure (w, t) of G_ℓ^+ by setting

$$w(y) = \begin{cases} |V(G_\ell^+)| - 1, & \text{if } y = x; \\ 1, & \text{otherwise,} \end{cases}$$

and $t = |V(G_\ell^+)|$. Indeed, if $D \subseteq V(G_\ell^+)$ is a total dominating set of G_ℓ^+ , then $\{x\}$ is a proper subset of D , which implies that $w(D) \geq |V(G_\ell^+)|$. Conversely, if $D \subseteq V(G_\ell^+)$ is a subset with $w(D) \geq |V(G_\ell^+)|$, then D contains x (since otherwise $w(D) \leq |V(G_\ell^+)| - 1$), moreover D must contain another vertex besides x , since otherwise $w(D) = |V(G_\ell^+)| - 1$. Hence, D is a total dominating set of G_ℓ^+ .

Case 2: $X \cap K_1 = \emptyset$.

In this case, every mtds of G_ℓ^+ is of the form $X \cup \{x\}$ for some $x \in K_1$. Therefore, we can obtain a total domishold structure (w, t) of G_ℓ^+ by setting

$$w(x) = \begin{cases} |K_1|, & \text{if } x \in X; \\ 1, & \text{if } x \in K_1; \\ 0, & \text{otherwise,} \end{cases}$$

and $t = w(X) + 1$. Indeed, if $D \subseteq V(G_\ell^+)$ is a total dominating set of G_ℓ^+ , then there exists a vertex $x \in K_1$ such that $\{x\} \cup X \subseteq D$, and therefore $w(D) \geq w(X) + 1$. Conversely, if $D \subseteq V(G_\ell^+)$ is a subset with $w(D) \geq w(X) + 1$, then $X \subseteq D$, since otherwise we would get the following contradictory chain of inequalities

$$t \leq w(D) \leq \sum_{x \in V(G_\ell^+)} w(x) - |K_1| = w(X) + |K_1| - |K_1| = w(X) < t.$$

Moreover, D must also contain a vertex from K_1 since otherwise $w(D) = w(X) < t$. Therefore, D is a total dominating set of G_ℓ^+ .

In each of the two cases, we showed that G_ℓ^+ is TD, contradicting Claim 1. This completes the proof of the claim. \square

In the proofs of the following claims, we will denote $S' = \cup_{v \in S} L_v$, for every $S \subseteq I$.

Claim 4. X is an independent set.

Proof. Suppose that there exists a pair of adjacent vertices in X . In particular, this implies that $X \cap K \neq \emptyset$. Let q denote the smallest index $i \in \{1, \dots, r\}$ such that $K_i \cap X \neq \emptyset$. Then $X \cap (\cup_{j < q} K_j) = \emptyset$, and set X dominates all vertices in the set $K \cup L \cup (\cup_{i \geq q} I_i)$.

Condition 3b implies that:

$$\text{For every } j < q \text{ and every } u \in I_j, \text{ it holds that } \ell(u) \leq 1. \quad (6)$$

Indeed, if $\ell(u) \geq 2$ for some $u \in I_j$ with $j < q$, then any $x \in K_q \cap X$ and a vertex $y \in I_q$ would form a pair of adjacent non-neighbors of u such that $\ell(u) \geq 2$ and $\ell(x) > 0$, contrary to condition 3b.

We consider two cases, depending on whether $\cup_{i < q} I_i \subseteq X$ or not.

Case 1: $\cup_{i < q} I_i \subseteq X$.

The set \mathcal{D} of all mtlds's of G_ℓ^+ can be partitioned into pairwise disjoint subsets

$$\mathcal{D} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_q,$$

where, for all $j = 1, \dots, q$,

$$\mathcal{D}_j = \{D \in \mathcal{D} \mid \min\{i \mid D \cap K_i \neq \emptyset\} = j\}.$$

Moreover, property (6) implies that for every $j \in \{1, \dots, q-1\}$, set \mathcal{D}_j consists of all sets D of the form

$$D = X \cup \{x\} \cup (\cup_{i < j} I'_i) \quad (7)$$

where $x \in K_j$, while the set \mathcal{D}_q consists of a unique set, namely $\mathcal{D}_q = \{X \cup (\cup_{i < q} I'_i)\}$.

Consider the weight function $w : V(G_\ell^+) \rightarrow \mathbb{Z}_+$ defined recursively as follows:

(1) For all $j = q-1, \dots, 1$, set

$$w(x) = \sum_{i=j+1}^{q-1} w(K_i) + 1 \text{ for all } x \in I'_j$$

(in particular, $w(x) = 1$ for all $x \in I'_{q-1}$), and

$$w(x) = \sum_{i=j}^{q-1} w(I'_i) \quad \text{for all } x \in K_j.$$

(2) For all $x \in X$, set $w(x) = \sum_{j < q} w(K_j) + 1$.

(3) For all other vertices x , set $w(x) = 0$.

We claim that the pair (w, t) , with w as above and $t = w(X) + \sum_{i < q} w(I'_i)$ is a total domishold structure of G_ℓ^+ .

Let D be a total dominating set in G_ℓ^+ . Then, D contains a mtlds in G_ℓ^+ , say D' . Then, either $D' = X \cup (\cup_{i < q} I'_i)$, or $D' = X \cup \{x\} \cup (\cup_{i < j} I'_i)$ where $1 \leq j < q$ and $x \in K_j$. In the former case, the weight of D' is equal to t , and hence $w(D) \geq t$. In the latter case, we also obtain

$$w(D') = w(X) + w(x) + \sum_{i < j} w(I'_i) = w(X) + \sum_{i=j+1}^{q-1} w(I'_i) + \sum_{i < j} w(I'_i) = t,$$

which again implies $w(D) \geq t$.

Conversely, let $D \subseteq V(G_\ell^+)$ be a set such that $w(D) \geq t$. Then, $X \subseteq D$, since otherwise we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - \sum_{j < q} w(K_j) - 1 \\ &= w(X) + \sum_{j < q} w(I'_j) + \sum_{j < q} w(K_j) - \sum_{j < q} w(K_j) - 1 \\ &< w(X) + \sum_{j < q} w(I'_j) = t. \end{aligned}$$

If $\cup_{j < q} I'_j \subseteq D$, then D is a total dominating set of G_ℓ^+ . So assume that $\cup_{j < q} I'_j \not\subseteq D$. Let $j \in \{1, \dots, q-1\}$ be the smallest index such that $I'_j \not\subseteq D$, and let $x \in I'_j \setminus D$. If $(\cup_{i \leq j} K_i) \cap D \neq \emptyset$, then D is a total dominating set of G_ℓ^+ . Suppose that $(\cup_{i \leq j} K_i) \cap D = \emptyset$. Then, we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - w(x) - \sum_{i \leq j} w(K_i) \\ &= w(X) + \sum_{i < q} w(I'_i) + \sum_{i < q} w(K_i) - \sum_{i=j+1}^{q-1} w(K_i) - 1 - \sum_{i \leq j} w(K_i) \\ &< w(X) + \sum_{i < q} w(I'_i) = t. \end{aligned}$$

Therefore, for all $D \subseteq V(G_\ell^+)$, we have $w(D) \geq t$ if and only if D is a total dominating set of G_ℓ^+ . This shows that G_ℓ^+ is TD in this case.

Case 2: $\cup_{i < q} I_i \not\subseteq X$.

Let $p \in \{1, \dots, q-1\}$ be the smallest index such that $I_p \not\subseteq X$. By Claim 3, $p \geq 2$. Let $x^* \in I_p \setminus X$. In this case, in order to dominate x^* , every mtds of G_ℓ^+ must contain a vertex from $\cup_{i \leq p} K_i$. Hence, the set \mathcal{D} of all mtds's of G_ℓ^+ can be partitioned into pairwise disjoint subsets

$$\mathcal{D} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_p,$$

where, for all $j = 1, \dots, p$,

$$\mathcal{D}_j = \{D \in \mathcal{D} \mid \min\{i \mid D \cap K_i \neq \emptyset\} = j\}.$$

Moreover, for every $j \in \{1, \dots, p\}$, set \mathcal{D}_j consists of all sets D of the form

$$D = X \cup \{x\} \cup (\cup_{i < j} I'_i) \tag{8}$$

where $x \in K_j$.

Consider the weight function $w : V(G_\ell^+) \rightarrow \mathbb{Z}_+$ defined recursively as follows:

(1) Set

$$w(x) = 1 \quad \text{for all } x \in K_p.$$

(2) For all $j = p-1, \dots, 1$, set

$$w(x) = \sum_{i=j+1}^p w(K_i) \text{ for all } x \in I'_j,$$

and

$$w(x) = \sum_{i=j}^{p-1} w(I'_i) + 1 \quad \text{for all } x \in K_j.$$

(3) For all $x \in X$, set $w(x) = \sum_{j \leq p} w(K_j)$.

(4) For all other vertices x , set $w(x) = 0$.

We claim that the pair (w, t) , with w as above and $t = w(X) + \sum_{i < p} w(I'_i) + 1$ is a total domishold structure of G_ℓ^+ .

Let D be a total dominating set in G_ℓ^+ . Then, D contains a mtds in G_ℓ^+ , say D' , of the form $D' = X \cup \{x\} \cup (\cup_{i < j} I'_i)$ where $1 \leq j \leq p$ and $x \in K_j$. Hence, the weight of D' is equal to

$$w(D') = w(X) + w(x) + \sum_{i < j} w(I'_i) = w(X) + \sum_{i=j}^{p-1} w(I'_i) + 1 + \sum_{i < j} w(I'_i) = t,$$

which implies $w(D) \geq t$.

Conversely, let $D \subseteq V(G_\ell^+)$ be a set such that $w(D) \geq t$. Then, $X \subseteq D$, since otherwise we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - \sum_{j \leq p} w(K_j) \\ &= w(X) + \sum_{j < p} w(I'_j) + \sum_{j \leq p} w(K_j) - \sum_{j \leq p} w(K_j) \\ &= w(X) + \sum_{j < p} w(I'_j) < t. \end{aligned}$$

Moreover, a similar reasoning shows that $(\cup_{i \leq p} K_i) \cap X \neq \emptyset$. Let $j \in \{1, \dots, p\}$ be the smallest index such that $D \cap K_j \neq \emptyset$. If $\cup_{i < j} I'_i \subseteq D$, then D is a total dominating set of G_ℓ^+ . So assume that $\cup_{i < j} I'_i \not\subseteq D$. Let $r \in \{1, \dots, j-1\}$ be an index such that $I'_r \not\subseteq D$, and let $x \in I'_r \setminus D$. Then, we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - w(x) - \sum_{i < j} w(K_i) \\ &= w(X) + \sum_{i < p} w(I'_i) + \sum_{i \leq p} w(K_i) - \sum_{i=r+1}^p w(K_i) - \sum_{i < j} w(K_i) \\ &\leq w(X) + \sum_{i < p} w(I'_i) < t. \end{aligned}$$

Therefore, for all $D \subseteq V(G_\ell^+)$, we have $w(D) \geq t$ if and only if D is a total dominating set of G_ℓ^+ . This shows that G_ℓ^+ is TD also in Case 2 and completes the proof of the claim. \square

Claim 5. $X \subseteq I$.

Proof. Suppose that $X \not\subseteq I$. By Claim 4, X contains at most one vertex from K . Since $X \not\subseteq I$, set X contains exactly one vertex, say x^* , from K . Let $q \in \{1, \dots, r\}$ such that $x^* \in K_q$. Since $I_1 \subseteq X$ and $X \cap K_1 = \emptyset$, we certainly have $q \geq 2$, and thus also $r \geq 2$. Also, $X \cap (\bigcup_{j \geq q} I_j) = \emptyset$ since otherwise X would not be independent, contradicting Claim 4. Let $p \in \{2, \dots, r\}$ be the smallest index such that $I_p \not\subseteq X$, then $p \leq q$ since $I_q \cap X = \emptyset$.

Condition 3b implies that:

$$\text{For every } j < p \text{ and every } u \in I_j, \text{ it holds that } \ell(u) \leq 1. \quad (9)$$

Indeed, if $\ell(u) \geq 2$ for some $u \in I_j$ with $j < p$, then x^* and a vertex $y \in I_q$ would form a pair of adjacent non-neighbors of u such that $\ell(u) \geq 2$ and $\ell(x^*) > 0$, contrary to condition 3b.

Taking $v \in I_p \setminus X$, we see that in order to dominate v , every total dominating set of G_ℓ^+ must contain a vertex from $N_{G_\ell^+}(v) = N_G(v) = \bigcup_{j=1}^p K_j$. Hence, the set \mathcal{D} of all mtds's of G_ℓ^+ can be partitioned into pairwise disjoint subsets

$$\mathcal{D} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_p,$$

where, for all $j = 1, \dots, p$,

$$\mathcal{D}_j = \{D \in \mathcal{D} \mid \min\{i \mid D \cap K_i \neq \emptyset\} = j\}.$$

We consider two cases according to the value of p .

Case 1: $p < q$.

We claim that in this case, for all $j = 1, \dots, p$, every $D \in \mathcal{D}_j$ is of the form

$$D = X \cup \{x\} \cup \bigcup_{i < j} I'_i \quad (10)$$

for some $x \in K_j$, where $I'_i = \bigcup_{v \in I_i} L_v$. On the one hand, it is straightforward to verify that every set D defined as in (10) is a dominating set in G_ℓ^+ . On the other hand, every set $D \in \mathcal{D}_j$ satisfies $|K_j \cap D| = 1$; this follows directly from the definition of \mathcal{D}_j . Moreover, the definition of \mathcal{D}_j implies that $D \cap (\bigcup_{i < j} K_i) = \emptyset$, which in turn implies that every vertex $v \in \bigcup_{i < j} I_i$ can only be dominated by its neighbor in L_v (by property (9), there is a unique element in L_v). Hence, it also holds that $\bigcup_{i < j} I'_i \subseteq D$ for every set $D \in \mathcal{D}_j$.

Consider the weight function $w : V(G_\ell^+) \rightarrow \mathbb{Z}_+$ defined recursively as follows:

- (1) For all $x \in K_p$, set $w(x) = 1$.
- (2) For all $j = p-1, p-2, \dots, 1$, set

$$w(x) = \sum_{i=j+1}^p w(K_i) \quad \text{for all } x \in I'_j$$

and

$$w(x) = \sum_{i=j}^{p-1} w(I'_i) + 1 \quad \text{for all } x \in K_j.$$

(3) For all $x \in X$, set $w(x) = \sum_{i=1}^p w(K_i)$.

(4) For all other vertices x , set $w(x) = 0$.

Let $I' := \cup_{i=1}^{p-1} I'_i$. We claim that the pair (w, t) , with w as above and $t = w(X) + w(I') + 1$, is a total domishold structure of G_ℓ^+ .

Let D be a total dominating set in G . Then, D contains a mtds in G , say D' , which is of the form

$$D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i$$

for some $x \in K_j$, where $1 \leq j \leq p$. We can bound the weight of D from below as

$$\begin{aligned} w(D) &\geq w(D') = w(X) + w(x) + \sum_{i < j} w(I'_i) \\ &= w(X) + \sum_{i=j}^{p-1} w(I'_i) + 1 + \sum_{i < j} w(I'_i) \\ &= w(X) + \sum_{i=1}^{p-1} w(I'_i) + 1 = t. \end{aligned}$$

Conversely, let $D \subseteq V(G_\ell^+)$ be a set such that $w(D) \geq t$. Then, $X \subseteq D$, since otherwise, assuming $x \in X \setminus D$, we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - w(x) \\ &= w(X) + \sum_{i \leq p} w(K_i) + \sum_{i < p} w(I'_i) - \sum_{i \leq p} w(K_i) \\ &= w(X) + \sum_{i < p} w(I'_i) < t. \end{aligned}$$

Moreover, D contains an element of $\cup_{i=1}^p K_i$ since otherwise we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - w(\cup_{i=1}^p K_i) \\ &= w(X) + \sum_{i \leq p} w(K_i) + \sum_{i < p} w(I'_i) - \sum_{i \leq p} w(K_i) \\ &= w(X) + \sum_{i < p} w(I'_i) < t. \end{aligned}$$

Let $j \in \{1, \dots, p\}$ be the smallest index such that $D \cap K_j \neq \emptyset$. Now, we will show that for every $i < j$ and every $v \in I_i$, set D contains the unique element of L_v . Suppose for a contradiction that $x \notin D$ where $L_v = \{x\}$ for some $v \in I_{i^*}$ and $i^* < j$. Then, we get the

following contradicting chain of inequalities

$$\begin{aligned}
t &\leq w(D) \leq w(V(G_\ell^+)) - \left(\sum_{i < j} w(K_i) + w(x) \right) \\
&= w(X) + \sum_{i \leq p} w(K_i) + \sum_{i < p} w(I'_i) - \sum_{i < j} w(K_i) - \sum_{i=i^*+1}^p w(K_i) \\
&\leq w(X) + \sum_{i < p} w(I'_i) < t.
\end{aligned}$$

Hence, we have proved that D contains a set of the form

$$D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i,$$

where $x \in K_j$ and $1 \leq j \leq p$. Therefore, D is a total dominating set of G_ℓ^+ .

Therefore, for all $D \subseteq V(G_\ell^+)$, we have $w(D) \geq t$ if and only if D is a total dominating set of G_ℓ^+ . This shows that G_ℓ^+ is TD, contrary to Claim 1, and completes the proof of Case 1.

Case 2: $p = q$.

In this case the set X dominates all vertices in $V(G_\ell^+) \setminus X$. The minimal dominating sets $D \in \mathcal{D}$ are of two types: for all $j = 1, \dots, p-1$, every $D \in \mathcal{D}_j$ is of the form

$$D = X \cup \{x\} \cup \bigcup_{i < j} I'_i$$

for some $x \in K_j$, and every $D \in \mathcal{D}_p = \mathcal{D}_q$ is of the form

$$D = X \cup \{y\} \cup \bigcup_{i < p} I'_i$$

for some $y \in A$, where $A := \left(\bigcup_{i=p}^r (K_i \cup I_i) \cup L_{x^*} \right) \setminus \{x^*\}$.

Consider the weight function $w : V(G_\ell^+) \rightarrow \mathbb{Z}_+$ defined recursively as follows:

- (1) For all $x \in A$, set $w(x) = 1$.
- (2) For all $j = p-1, p-2, \dots, 1$, set

$$w(x) = \sum_{i=j+1}^{p-1} w(K_i) + w(A) \quad \text{for all } x \in I'_j$$

(in particular, $w(x) = w(A)$ for all $x \in I'_{p-1}$), and

$$w(x) = \sum_{i=j}^{p-1} w(I'_i) + 1 \quad \text{for all } x \in K_j,$$

(3) For all $x \in X$, set $w(x) = w(A) + w(B)$ where $B := \bigcup_{i < p} K_i = K \setminus (A \cup \{x^*\})$.

Let $I' := \bigcup_{i=1}^{p-1} I'_i$. We claim that the pair (w, t) , with w as above and $t = w(X) + w(I') + 1$, is a total domishold structure of G_ℓ^+ .

Let D be a total dominating set in G . Then, D contains a mtds in G , say D' . Suppose that D' is of the form $D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i$ for some $x \in K_j$ and $1 \leq j \leq p-1$. Then, we can bound the weight of D from below as

$$\begin{aligned} w(D) &\geq w(D') = w(X) + w(x) + \sum_{i < j} w(I'_i) \\ &= w(X) + \sum_{i=j}^{p-1} w(I'_i) + 1 + \sum_{i < j} w(I'_i) = t. \end{aligned}$$

Similarly, if D' is of the form $D' = X \cup \{y\} \cup \bigcup_{i < p} I'_i$ for some $y \in A$, we get

$$\begin{aligned} w(D) &\geq w(D') = w(X) + w(y) + \sum_{i < j} w(I'_i) \\ &= w(X) + 1 + \sum_{i < p} w(I'_i) = t. \end{aligned}$$

Conversely, let $D \subseteq V(G_\ell^+)$ be a set such that $w(D) \geq t$. Then, $X \subseteq D$, since otherwise, assuming $x \in X \setminus D$, we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - w(x) \\ &= w(X) + w(A) + w(B) + w(I') - (w(A) + w(B)) \\ &= w(X) + w(I') < t. \end{aligned}$$

A similar reasoning shows that D contains an element of $A \cup B$.

Suppose first that $D \cap B = \emptyset$. Then there exists a vertex $y \in A \cap D$, and we only need to show that $I' \subseteq D$. Suppose for a contradiction that $x \in I' \setminus D$. Then $w(x) \geq w(A)$, and we get the following contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - \left(\sum_{i < p} w(K_i) + w(x) \right) \\ &\leq w(X) + w(A) + w(B) + w(I') - w(B) - w(A) \\ &= w(X) + w(I') < t. \end{aligned}$$

Now, suppose that D contains an element of B , and let $j \in \{1, \dots, p-1\}$ be the smallest index such that $D \cap K_j \neq \emptyset$. We will show that $\bigcup_{i < j} I'_i \subseteq D$. Suppose that $x \notin D$ where $x \in L_v$ for some $v \in I_{i^*}$ and $i^* < j$. Then, we get the following contradicting chain of

inequalities

$$\begin{aligned}
t &\leq w(D) \leq w(V(G_\ell^+)) - \left(\sum_{i < j} w(K_i) + w(x) \right) \\
&= w(X) + w(A) + w(B) + w(I') - \sum_{i < j} w(K_i) - \sum_{i=i^*+1}^{p-1} w(K_i) - w(A) \\
&\leq w(X) + w(I') < t.
\end{aligned}$$

Hence, we have proved that D contains a minimal total dominating set, and thus D is a total dominating set of G_ℓ^+ .

Therefore, for all $D \subseteq V(G_\ell^+)$, we have $w(D) \geq t$ if and only if D is a total dominating set of G_ℓ^+ . This shows that G_ℓ^+ is TD, contrary to Claim 1. This completes the proof of Case 2, and with it the proof of the claim. \square

Claim 6. $X \neq I$.

Proof. Suppose for a contradiction that $X = I$. In this case, X dominates all vertices of $V(G_\ell^+) \setminus X$. The set \mathcal{D} of all mtds's of G_ℓ^+ can be partitioned into pairwise disjoint subsets

$$\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \dots \cup \mathcal{D}_r,$$

where

$$\mathcal{D}_0 = \{D \in \mathcal{D} \mid D \cap K = \emptyset\},$$

and for all $j = 1, \dots, r$,

$$\mathcal{D}_j = \{D \in \mathcal{D} \mid \min\{i \mid D \cap K_i \neq \emptyset\} = j\}.$$

Condition 3b implies that:

$$\text{For every } j < r \text{ and every } u \in I_j, \text{ it holds that } \ell(u) = 1. \quad (11)$$

Indeed, we have $\ell(u) \geq 1$ since $I \subseteq X$, however, if $\ell(u) \geq 2$ for some $u \in I_j$ with $j < r$, then any vertex $x \in K_r$ and a vertex $y \in I_r$ would form a pair of adjacent non-neighbors of u such that $\ell(u) \geq 2$ and $\ell(y) > 0$, contrary to condition 3b.

It follows that for all $j = 1, \dots, r$, every mtds $D \in \mathcal{D}_j$ is of the form

$$D = X \cup \{x\} \cup \bigcup_{i < j} I'_i \quad (12)$$

for some $x \in K_j$.

Condition 3a implies that there can be at most one u vertex in I with $\ell(u) \geq 2$; moreover, if such a vertex exists, condition (11) implies that it belongs to I_r . Let x^+ be any vertex v in I_r with maximum value of $\ell(v)$, and let $d = \ell(x^+)$. Then, there are exactly d sets $D \in \mathcal{D}_0$; they are all of the form

$$D = X \cup \bigcup_{v \in I \setminus \{x^+\}} L_v \cup \{\ell_{x^+}\} \quad (13)$$

where $\ell_{x^+} \in L_{x^+}$.

Consider the weight function $w : V(G_\ell^+) \rightarrow \mathbb{Z}_+$ defined recursively as follows:

- (1) For all $x \in L_{x^+}$, set $w(x) = 1$.
- (2) For all $x \in I'_r \setminus L_{x^+}$, set $w(x) = d$.
- (3) For all $j = r, \dots, 1$, set

$$w(x) = \sum_{i=j}^r w(I'_i) - d + 1 \quad \text{for all } x \in K_j.$$

- (4) For all $j = r-1, \dots, 1$, set

$$w(x) = \sum_{i>j} w(K_i) + d \quad \text{for all } x \in I'_j.$$

- (5) For all $x \in X = I$, set $w(x) = w(K) + d$.

We claim that the pair (w, t) with w as above and $t = w(X) + w(I') - d + 1$, where $I' = \cup_{i \leq r} I'_i$, is a total domishold structure of G_ℓ^+ .

Let D be a total dominating set in G . Then, D contains a mtds D' where $D' \in \mathcal{D}_j$ for some $j \in \{0, 1, \dots, r\}$. If $j = 0$ then D' is of the form given by (13): $D' = X \cup \bigcup_{v \in I \setminus \{x^+\}} L_v \cup \{\ell_{x^+}\}$ for some $\ell_{x^+} \in L_{x^+}$. We can bound the weight of D from below as

$$\begin{aligned} w(D) &\geq w(D') = w(X) + \sum_{v \in I \setminus \{x^+\}} w(L_v) + w(\ell_{x^+}) \\ &= w(X) + \sum_{i=j}^r w(I'_i) - w(L_{x^+}) + w(\ell_{x^+}) \\ &= w(X) + w(I') - d + 1 = t. \end{aligned}$$

Similarly, if $j > 0$, then D' is of the form given by (12): $D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i$ for some $x \in K_j$, and we can bound the weight of D from below as

$$\begin{aligned} w(D) &\geq w(D') = w(X) + w(x) + \sum_{i < j} w(I'_i) \\ &= w(X) + \sum_{i=j}^r w(I'_i) - d + 1 + \sum_{i < j} w(I'_i) \\ &= w(X) + w(I') - d + 1 = t. \end{aligned}$$

Conversely, let $D \subseteq V(G_\ell^+)$ be a set such that $w(D) \geq t$. Then, $X \subseteq D$, since otherwise, assuming $x \in X \setminus D$, we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - w(x) \\ &= w(X) + w(K) + w(I') - (w(K) + d) \\ &= w(X) + w(I') - d < t. \end{aligned}$$

Suppose first that $D \cap K \neq \emptyset$, and let $j \in \{1, \dots, r\}$ be the smallest index such that $D \cap K_j \neq \emptyset$. Now, we will show that for every $i < j$ and every $v \in I_i$, set D contains the unique element of L_v . Suppose for a contradiction that $x \notin D$ where $L_v = \{x\}$ for some $v \in I_{i^*}$ and $i^* < j$. Then, we get the following contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - \left(\sum_{i < j} w(K_i) + w(x) \right) \\ &= w(X) + w(K) + w(I') - \sum_{i < j} w(K_i) - \sum_{i > i^*} w(K_i) - d \\ &\leq w(X) + w(I') - d < t. \end{aligned}$$

Hence, we have proved that D contains a set of the form

$$D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i,$$

where $x \in K_j$. Therefore, D is a total dominating set of $G^{+\ell}$.

Suppose now that $D \cap K = \emptyset$. Then, it is enough to show that D contains one element of L_v for each $v \in I$. Suppose this is not the case, and let $v \in I$ be a vertex such that $D \cap L_v = \emptyset$. The definition of w implies that $w(L_v) \geq d$, and so we get the following contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(V(G_\ell^+)) - \left(w(K) + w(L_v) \right) \\ &= w(X) + w(K) + w(I') - w(K) - w(L_v) \\ &\leq w(X) + w(I') - d < t. \end{aligned}$$

Hence, D contains a set of the form given by (13), and it is therefore a total dominating set of G_ℓ^+ .

Therefore, for all $D \subseteq V(G_\ell^+)$, we have $w(D) \geq t$ if and only if D is a total dominating set of G_ℓ^+ . This shows that G_ℓ^+ is TD, contrary to Claim 1. This completes the proof of the claim. \square

Let p be defined as the minimum index $j \in \{1, \dots, r\}$ such that $I_j \not\subseteq X$. Notice that by Claim 6, p is well defined, and by Claim 3, $p \geq 2$. In particular, this implies that $r \geq 2$.

Condition 3c implies that:

$$\text{For every } j < p - 1 \text{ and every } u \in I_j, \text{ it holds that } \ell(u) = 1. \quad (14)$$

Indeed, the definition of p implies $\ell(u) \geq 1$. Moreover, if $\ell(u) \geq 2$ for some $u \in I_j$ with $j < p - 1$, then any triple $\{x, y, z\}$ with $x \in I_{p-1}$, $y \in K_{p-1}$ and $z \in I_p$ forms an induced P_3 in $G - N_G[u]$, contrary to condition 3c.

Every total dominating set in G_ℓ^+ must contain a member of $\cup_{j \leq p} K_j$, in order to contain a neighbor of a vertex in $I_p \setminus X$. The set \mathcal{D} of all mtds's of G_ℓ^+ can thus be partitioned into pairwise disjoint subsets

$$\mathcal{D} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_p,$$

where for all $j = 1, \dots, p$,

$$\mathcal{D}_j = \{D \in \mathcal{D} \mid \min\{i \mid D \cap K_i \neq \emptyset\} = j\}.$$

For $1 \leq j \leq p-1$, every $D \in \mathcal{D}_j$ is of the form

$$D = X \cup \{x\} \cup \bigcup_{i < j} I'_i \quad (15)$$

where $x \in K_j$.

Claim 7. $X \cap (\bigcup_{j \geq p} I_j) = \emptyset$.

Proof. Suppose for a contradiction that X contains a vertex from $\bigcup_{j \geq p} I_j$. Then, equation (15) also holds for every $D \in \mathcal{D}_j$ for $j = p$, since vertex $x \in K_p$ will already be dominated by some vertex in X .

In this case, condition (14) can be further strengthened to include the case $j = p-1$:

For every $j < p$ and every $u \in I_j$, it holds that $\ell(u) = 1$.

Indeed, the definition of p implies $\ell(u) \geq 1$. Moreover, if $\ell(u) \geq 2$ for some $u \in I_j$ with $j < p$, then any vertex $x \in X \cap (\bigcup_{j \geq p} I_j)$ and a vertex $y \in K_p$ would form a pair of adjacent non-neighbors of u such that $\ell(u) \geq 2$ and $\ell(x) > 0$, contrary to condition 3b.

Consider the weight function $w : V(G_\ell^+) \rightarrow \mathbb{Z}_+$ defined recursively as follows:

(1) For all $x \in K_p$, set $w(x) = 1$.

(2) For all $j = p-1, \dots, 1$, set

$$w(x) = \sum_{i=j+1}^p w(K_i) \quad \text{for all } x \in I'_j$$

and

$$w(x) = \sum_{i=j}^{p-1} w(I'_i) + 1 \quad \text{for all } x \in K_j.$$

(3) For all $x \in X$, set $w(x) = \sum_{i=1}^p w(K_i)$.

(4) For all other vertices x , set $w(x) = 0$.

Notice that $w(x) = w(K)$ for all $x \in X$. We claim that the pair (w, t) with w as above and $t = w(X) + \sum_{j < p} w(I'_j) + 1$ is a total domishold structure of G_ℓ^+ .

Let D be a total dominating set in G_ℓ^+ . Then, D contains a mtlds in G_ℓ^+ , say D' , which is of the form

$$D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i \quad (16)$$

for some $x \in K_j$, where $1 \leq j \leq p$. We can bound the weight of D from below as

$$\begin{aligned}
w(D) &\geq w(D') = w(X) + w(x) + \sum_{i < j} w(I'_i) \\
&= w(X) + \sum_{i=j}^{p-1} w(I'_i) + 1 + \sum_{i < j} w(I'_i) \\
&= w(X) + \sum_{i < p} w(I'_i) + 1 = t.
\end{aligned}$$

Conversely, let $D \subseteq V(G_\ell^+)$ be a set such that $w(D) \geq t$. Then, $X \subseteq D$, since otherwise we get a contradicting chain of inequalities

$$\begin{aligned}
t &\leq w(D) \leq w(X) + w(K) + \sum_{j < p} w(I'_j) - w(K) \\
&= w(X) + \sum_{j < p} w(I'_j) < t.
\end{aligned}$$

Similarly, since $w(\cup_{i=1}^p K_i) = w(K)$, we infer that D must contain an element from $\cup_{i=1}^p K_i$. Let $j \in \{1, \dots, p\}$ be the smallest index such that $D \cap K_j \neq \emptyset$. Notice that for every $x \in \cup_{i < j} I'_i$, we have $w(x) \geq \sum_{i=j}^p w(K_i)$. Therefore, $\cup_{i < j} I'_i \subseteq D$ since otherwise we get the following contradicting chain of inequalities

$$\begin{aligned}
t &\leq w(D) \leq w(X) + w(K) + \sum_{j < p} w(I'_j) - \sum_{j < j} w(K_i) - \sum_{i=j}^p w(K_i) \\
&= w(X) + \sum_{j < p} w(I'_j) < t.
\end{aligned}$$

Hence, we have proved that D contains a set of the form

$$D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i,$$

where $x \in K_j$ and $1 \leq j \leq p$. Therefore, D is a total dominating set of $G^+ \ell$.

Therefore, for all $D \subseteq V(G_\ell^+)$, we have $w(D) \geq t$ if and only if D is a total dominating set of G_ℓ^+ . This shows that G_ℓ^+ is TD and completes the proof of the claim. \square

By Claim 7, we have $X \cap (\cup_{j \geq p} I_j) = \emptyset$. Therefore, every $D \in \mathcal{D}_p$ is of the form

$$D = X \cup \{x\} \cup \bigcup_{i < p-1} I'_i \cup \bigcup_{v \in I_{p-1}} \{\ell_v\} \cup \{y\}$$

where $x \in K_p$, $\ell_v \in L_v$ for all $v \in I_{p-1}$, and $y \in (\cup_{j \geq p} (K_j \cup I_j)) \setminus \{x\}$. (The additional vertex y is needed to dominate x .)

Let $A = \cup_{j \geq p} (K_j \cup I_j) \setminus K_p$, let x^+ be any vertex v in I_{p-1} that maximizes $\ell(v)$, and let $d = \ell(x^+)$. Also, let $a = |A|$ and $k = |K_p|$.

Claim 8. $d = 1$.

Proof. Suppose that $d > 1$. Then, $k = a = 1$. Indeed, if $k \geq 2$, then any triple $\{x, y, z\}$ with $x, y \in K_p$, $x \neq y$, and $z \in I_p$ forms a (non-induced) P_3 in $G - N_G[x^+]$, contrary to condition 3c (note that $\ell(x^+) = d \geq 2$). Similarly, if $a \geq 2$, then any triple $\{x, y, z\}$ with $x, y \in A$, $x \neq z$, and $y \in K_p$ forms a (not necessarily induced) P_3 in $G - N_G[x^+]$, again contradicting condition 3c.

In particular, $a = 1$ implies that $A = I_p$, and $r = p$. Consider the weight function $w : V(G_\ell^+) \rightarrow \mathbb{Z}_+$ defined recursively as follows:

- (1) For all $x \in L_{x^+}$, set $w(x) = 1$.
- (2) For all $x \in (I'_{p-1} \setminus L_{x^+}) \cup A$, set $w(x) = d$.
- (3) For the unique $x \in K_p$, set $w(x) = d + 1$.
- (4) For all $j = p - 1, \dots, 1$, set

$$w(x) = \sum_{i=j}^{p-1} w(I'_i) + d + 2 \quad \text{for all } x \in K_j.$$

- (5) For all $j = p - 2, \dots, 1$, set

$$w(x) = \sum_{i=j+1}^p w(K_i) - 1 \quad \text{for all } x \in I'_j, \text{ and}$$

- (6) For all $x \in X$, set $w(x) = w(K) - 1$.
- (7) For all other vertices x (that is, for all $x \in I_{p-1}$), set $w(x) = 0$.

We claim that the pair (w, t) with w as above and $t = w(X) + w(I') + d + 2$, where $I' = \cup_{i < p} I'_i$, is a total domishold structure of G_ℓ^+ .

Let D be a total dominating set in G_ℓ^+ . Then, D contains a mtds D' in G_ℓ^+ where $D' \in \mathcal{D}_j$ for some $j \in \{1, \dots, p\}$. If $j = p$ then

$$D' = X \cup K_p \cup \bigcup_{v \in I \setminus \{x^+\}} L_v \cup \{\ell_{x^+}\} \cup A$$

where $\ell_{x^+} \in L_{x^+}$. Since $w(\bigcup_{v \in I \setminus \{x^+\}} L_v \cup \{\ell_{x^+}\}) = \sum_{i < p} w(I'_i) - (d - 1)$, we can bound the weight of D from below as

$$\begin{aligned} w(D) &\geq w(D') = w(X) + w(K_p) + \sum_{i < p} w(I'_i) - (d - 1) + w(A) \\ &\geq w(X) + d + 1 + w(I') - d + 1 + d = t. \end{aligned}$$

Similarly, if $j < p$, then

$$D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i$$

where $x \in K_j$. In this case, we can bound the weight of D from below as

$$\begin{aligned}
w(D) &\geq w(D') = w(X) + w(x) + \sum_{i < j} w(I'_i) \\
&\geq w(X) + \sum_{i=j}^{p-1} w(I'_i) + d + 2 + \sum_{i < j} w(I'_i) \\
&= w(X) + w(I') + d + 2 = t.
\end{aligned}$$

Conversely, let $D \subseteq V(G_\ell^+)$ be a set such that $w(D) \geq t$. Then, $X \subseteq D$, since otherwise we get a contradicting chain of inequalities

$$\begin{aligned}
t &\leq w(D) \leq w(X) + w(I') + w(K) + w(A) - (w(K) - 1) \\
&= w(X) + w(I') + d + 1 < t.
\end{aligned}$$

Similarly, D must contain an element from K , since otherwise we get a contradicting chain of inequalities

$$\begin{aligned}
t &\leq w(D) \leq w(X) + w(I') + w(A) \\
&= w(X) + w(I') + d < t.
\end{aligned}$$

Let $j \in \{1, \dots, p\}$ be the smallest index such that $D \cap K_j \neq \emptyset$. Suppose first that $j = p$. In this case, $A \subseteq D$, since otherwise we get a contradicting chain of inequalities

$$\begin{aligned}
t &\leq w(D) \leq w(X) + w(I') + w(K_p) \\
&= w(X) + w(I') + d + 1 < t.
\end{aligned}$$

Notice that $w(L_v) \geq d$ for all $v \in \cup_{i < p} I_i$. This implies that for all $v \in \cup_{i < p} I_i$, we have $L_v \cap D \neq \emptyset$, since otherwise we get a contradicting chain of inequalities

$$\begin{aligned}
t &\leq w(D) \leq w(X) + w(I') + w(K_p) + w(A) - w(L_v) \\
&\leq w(X) + w(I') + d + 1 + d - d < t.
\end{aligned}$$

Hence, if $j = p$ then D contains a set of the form $D' = X \cup K_p \cup \bigcup_{v \in I \setminus \{x^+\}} L_v \cup \{\ell_{x^+}\} \cup A$ where $\ell_{x^+} \in L_{x^+}$, and is therefore a total dominating set in G_ℓ^+ .

Suppose now that $j < p$. Notice that $w(x) \geq \sum_{i=j}^p w(K_i) - 1$ holds for all $x \in \cup_{i < j} I'_i$. This implies that $\cup_{i < j} I'_i \subseteq X$, since otherwise we get

$$\begin{aligned}
t &\leq w(D) \leq w(X) + w(I') + \sum_{i=j}^p w(K_i) + w(A) - \left(\sum_{i=j}^p w(K_i) - 1 \right) \\
&= w(X) + w(I') + d + 1 < t.
\end{aligned}$$

Hence, if $j < p$ then D contains a set of the form $D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i$ where $x \in K_j$, and is therefore a total dominating set in G_ℓ^+ .

Therefore, for all $D \subseteq V(G_\ell^+)$, we have $w(D) \geq t$ if and only if D is a total dominating set of G_ℓ^+ . This shows that G_ℓ^+ is TD and completes the proof of the claim. \square

By Claim 8, it holds that $d = 1$. Recall that $a = |A|$ and $k = |K_p|$. Consider the weight function $w : V(G_\ell^+) \rightarrow \mathbb{Z}_+$ defined recursively as follows:

- (1) For all $x \in A$, set $w(x) = 1$.
- (2) For all $x \in K_p$, set $w(x) = a + 1$.
- (3) For all $x \in I'_{p-1}$, set $w(x) = k(a + 1) - 1$.
- (4) For all $j = p - 1, \dots, 1$, set

$$w(x) = \sum_{i=j}^{p-1} w(I'_i) + a + 2 \quad \text{for all } x \in K_j.$$

- (5) For all $j = p - 2, \dots, 1$, set

$$w(x) = \sum_{i=j+1}^p w(K_i) - 1 \quad \text{for all } x \in I'_j, \text{ and}$$

- (6) For all $x \in X$, set $w(x) = w(K \setminus A) - 1$.
- (7) For all other vertices x (that is, for all $x \in I_{p-1}$), set $w(x) = 0$.

We claim that the pair (w, t) with w as above and $t = w(X) + w(I') + a + 2$, where $I' = \cup_{i < p} I'_i$, is a total domishold structure of G_ℓ^+ .

Let D be a total dominating set in G_ℓ^+ . Then, D contains a mtds D' in G_ℓ^+ where $D' \in \mathcal{D}_j$ for some $j \in \{1, \dots, p\}$. If $j = p$ then

$$D' = X \cup \{x\} \cup \bigcup_{i < p} I'_i \cup \{y\}$$

where $x \in K_p$ and $y \in (\cup_{j \geq p} (K_j \cup I_j)) \setminus \{x\}$. Since $w(y) \geq 1$, we can bound the weight of D from below as

$$\begin{aligned} w(D) &\geq w(D') = w(X) + w(x) + \sum_{i < p} w(I'_i) + w(y) \\ &\geq w(X) + a + 1 + w(I') + 1 = t. \end{aligned}$$

Similarly, if $j < p$, then

$$D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i$$

where $x \in K_j$. In this case, we can bound the weight of D from below as

$$\begin{aligned} w(D) &\geq w(D') = w(X) + w(x) + \sum_{i < j} w(I'_i) \\ &\geq w(X) + \sum_{i=j}^{p-1} w(I'_i) + a + 2 + \sum_{i < j} w(I'_i) \\ &= w(X) + w(I') + a + 2 = t. \end{aligned}$$

Conversely, let $D \subseteq V(G_\ell^+)$ be a set such that $w(D) \geq t$. Then, $X \subseteq D$, since otherwise we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(X) + w(I') + w(K \setminus A) + w(A) - (w(K \setminus A) - 1) \\ &= w(X) + w(I') + a + 1 < t. \end{aligned}$$

Similarly, D must contain an element from $K \setminus A = \cup_{j \leq p} K_j$, since otherwise we get a contradicting chain of inequalities

$$\begin{aligned} t &\leq w(D) \leq w(X) + w(I') + w(K \setminus A) + w(A) - w(K \setminus A) \\ &= w(X) + w(I') + a < t. \end{aligned}$$

Let $j \in \{1, \dots, p\}$ be the smallest index such that $D \cap K_j \neq \emptyset$. Suppose first that $j = p$. In this case, $|D \cap (A \cup K_p)| \geq 2$, since otherwise, denoting by x the unique element in $D \cap (A \cup K_p)$, we get

$$\begin{aligned} t &\leq w(D) \leq w(X) + w(I') + w(x) \\ &= w(X) + w(I') + a + 1 < t. \end{aligned}$$

Notice that $w(x) \geq k(a+1) - 1$ holds for all $x \in I'$. This implies that $I' \subseteq D$, since otherwise we get

$$\begin{aligned} t &\leq w(D) \leq w(X) + w(I') + w(K_p) + w(A) - (k(a+1) - 1) \\ &= w(X) + w(I') + k(a+1) + a - k(a+1) + 1 < t. \end{aligned}$$

Hence, if $j = p$ then D contains a set of the form $D' = X \cup \{x\} \cup \bigcup_{i < p} I'_i \cup \{y\}$ where $x \in K_p$ and $y \in (\cup_{j \geq p} (K_j \cup I_j)) \setminus \{x\}$, and is therefore a total dominating set in G_ℓ^+ . Suppose now that $j < p$. Notice that $w(x) \geq \sum_{i=j}^p w(K_i) - 1$ holds for all $x \in \cup_{i < j} I'_i$. This implies that $\cup_{i < j} I'_i \subseteq X$, since otherwise we get

$$\begin{aligned} t &\leq w(D) \leq w(X) + w(I') + \sum_{i=j}^p w(K_i) + w(A) - \left(\sum_{i=j}^p w(K_i) - 1 \right) \\ &= w(X) + w(I') + a + 1 < t. \end{aligned}$$

Hence, if $j < p$ then D contains a set of the form $D' = X \cup \{x\} \cup \bigcup_{i < j} I'_i$ where $x \in K_j$ and is therefore a total dominating set in G_ℓ^+ .

Therefore, for all $D \subseteq V(G_\ell^+)$, we have $w(D) \geq t$ if and only if D is a total dominating set of G_ℓ^+ . This shows that G_ℓ^+ is TD and completes the proof of the theorem. \square

5 Algorithmic Aspects

In this section, we show that total domishold graphs can be recognized in polynomial time, and examine some algorithmic consequences of this result. A polynomial time algorithm for the recognition of total domishold graphs will be obtained by reducing the problem to the problem of recognizing threshold Boolean functions given by a positive DNF.

Theorem 6. *There exists a polynomial time algorithm for recognizing total domishold graphs. If the input graph G is total domishold, the algorithm also computes an integral total domishold structure of G .*

Proof. Theorem 2 and Lemma 2 imply that the following polynomial time algorithm correctly determines whether G is total domishold, and if this is the case, computes a total domishold structure of it. First, compute the complete DNF ϕ of the neighborhood function f_G of G . More specifically, let $\phi = \bigvee_{S \in \mathcal{N}} \bigwedge_{u \in S} x_u$ where \mathcal{N} is the set of neighborhoods of vertices of G that do not properly contain any other neighborhood. Second, apply the algorithm given by Theorem 2 to the input ϕ . If the algorithm detects that f_G is not threshold, then G is not total domishold. Otherwise, the algorithm will compute an integral separating structure (w_1, \dots, w_n, t) of f_G , in which case Lemma 2 implies that $(w; \sum_{i=1}^n w_i - t)$ with $w(v_i) = w_i$ for all $i \in [n]$ is a total domishold structure of G . \square

Let us now examine some consequences of Theorem 6. The *total domination problem* is the problem of finding a minimum-sized total dominating set in a given graph. The problem is NP-complete in general, and remains NP-complete even for restricted graph classes such as bipartite graphs or split graphs (see [6]). On the positive side, polynomial time algorithms have been designed for several graph classes (see, e.g., [14] for an overview). With the exception of dually chordal graphs [3], all polynomial time algorithms for the total dominating set problem available in the literature (we are aware of) deal with hereditary graph classes. The following result provides another example of a non-hereditary graph class for which the problem is polynomial.

Proposition 6. *The total dominating set problem is solvable in polynomial time for total domishold graphs.*

Proof. Applying Theorem 6, we may assume that the input graph G is given together with an integral total domishold structure (w, t) . A greedy approach can be now used to find a minimum-sized total dominating set S : Start with the empty set, $S = \emptyset$, and, as long as $w(S) < t$, keep adding to S vertices according to non-increasing weights. The correctness and the polynomial running time of this algorithm are immediate. \square

One can similarly show that the more general problem, in which the input graph is equipped with a given cost function on the vertices and the task is to compute a total dominating set of minimum cost can be solved in pseudopolynomial time by a dynamic programming algorithm.

While the total dominating set problem is NP-complete for chordal graphs (in fact, even for split graphs), Proposition 6 shows that the problem is polynomial in the class of HTD graphs, which, by Corollary 4, is a subclass of chordal graphs.

6 Conclusion

In conclusion, we would like to mention the following open questions related to total domishold graphs. Is every $\{F_1, \dots, F_{13}\}$ -free graph total domishold? What is the complexity of recognizing HTD graphs?

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